Nodal, edge, and face lowest order virtual elements: exact sequences & interpolation estimates

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NEMESIS Workshop

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Aim of the talk

$$H^1 \xrightarrow{\nabla} H(\nabla\times) \xrightarrow{\nabla\times} H(\nabla\cdot) \xrightarrow{\nabla\cdot} L^2$$

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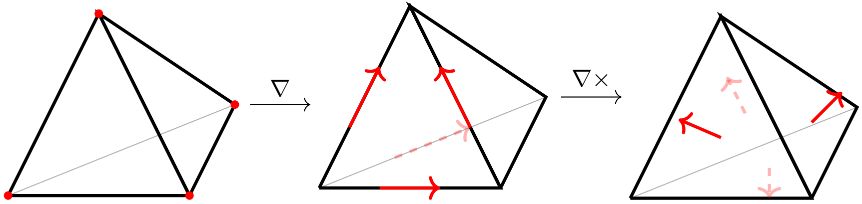

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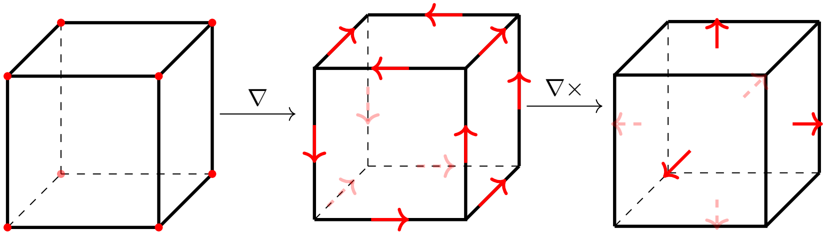

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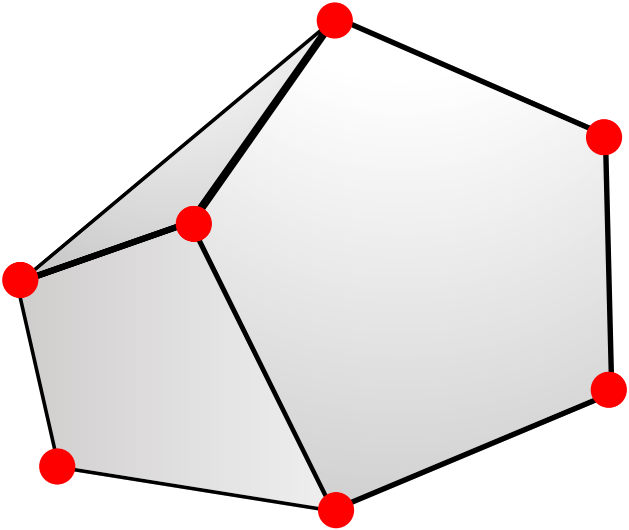

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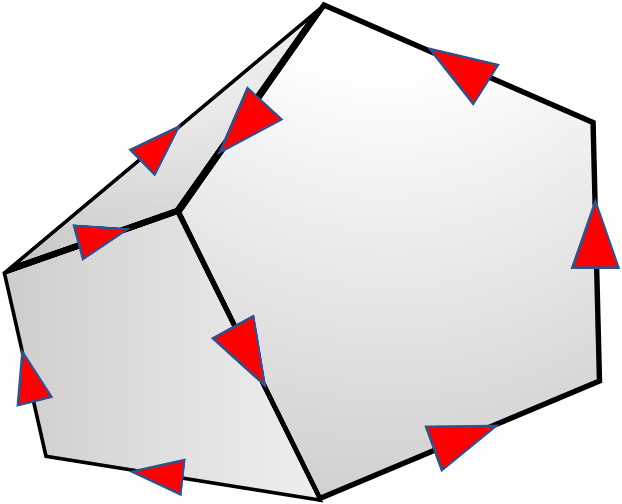

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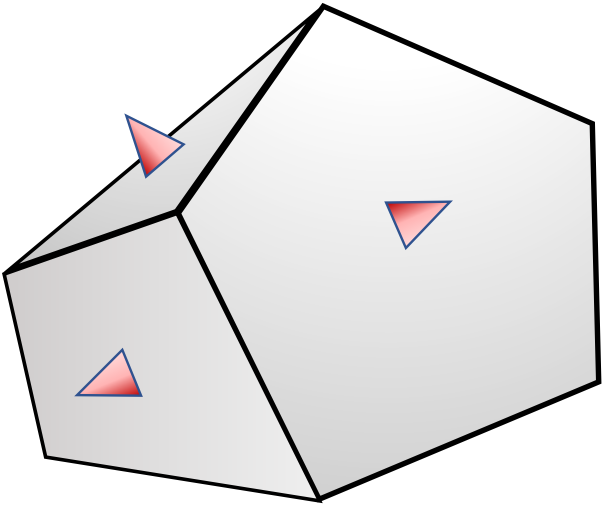

Aim of the talk

- why and how?

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- recall nodal-edge-face sequence for 3D VEM
  [Beirao da Veiga, Brezzi, Dassi, Marini, Russo, CMAME 2018, SINUM 2018]

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- why and how?
- recall nodal-edge-face sequence for 3D VEM
  [Beirao da Veiga, Brezzi, Dassi, Marini, Russo, CMAME 2018, SINUM 2018]
- interpolation estimates in the three spaces

Why and how?

Possible applications: Maxwell equations

$$
\begin{cases}
\varepsilon \mathbf{E}_t + \sigma \mathbf{E} - \nabla\times(\mu^{-1}\mathbf{B}) = \mathbf{J} & \text{in } \Omega,\ \forall t \in (0,T] \\
\mathbf{B}_t + \nabla\times \mathbf{E} = \mathbf{0} & \text{in } \Omega,\ \forall t \in (0,T] \\
\mathbf{E}(0) = \mathbf{E}^0, \quad \mathbf{B}(0) = \mathbf{B}^0 & \text{in } \Omega \\
\mathbf{E} \times \mathbf{n}_\Omega = 0, \quad \mathbf{B}\cdot\mathbf{n}_\Omega = 0 & \text{on } \partial\Omega
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$$

It suffices to take the $\nabla\cdot$ of the second equation

Possible applications: MHD models

$$\begin{cases} \mathbf{u}_t + (\nabla \mathbf{u})\mathbf{u} - Re^{-1}\Delta \mathbf{u} - s\,\mathbf{j} \times \mathbf{B} + \nabla p & = \mathbf{f} & \text{in } \Omega \\ \mathbf{j} - Re_m^{-1}\,\nabla \times \mathbf{B} & = \mathbf{0} & \text{in } \Omega \\ \mathbf{B}_t + \nabla \times \mathbf{E} & = \mathbf{0} & \text{in } \Omega \\ \nabla \cdot \mathbf{B} & = 0 & \text{in } \Omega \\ \nabla \cdot \mathbf{u} & = 0 & \text{in } \Omega \end{cases}$$

where

$$\mathbf{j} := \mathbf{E} + \mathbf{u} \times \mathbf{B}$$

Coupling of the FEM and the VEM

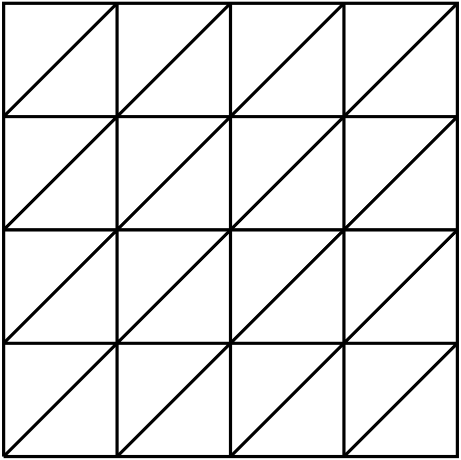
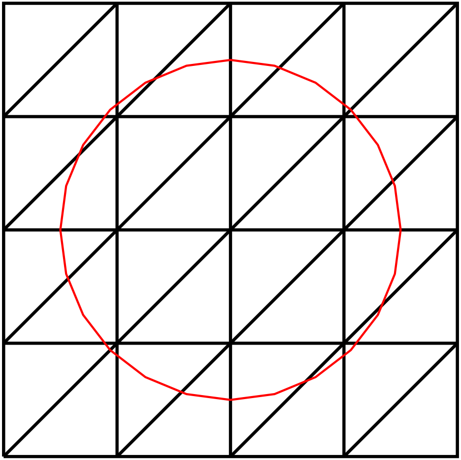


Figure: Imagine everything in 3D

Coupling of the FEM and the VEM



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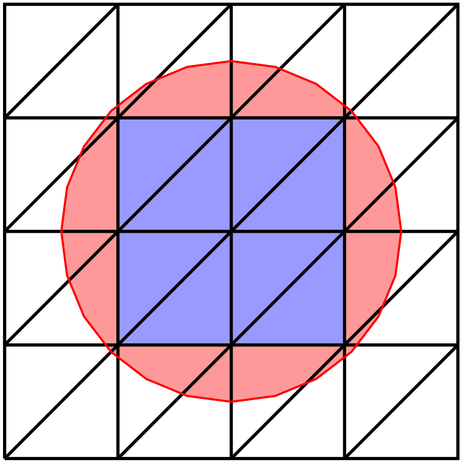
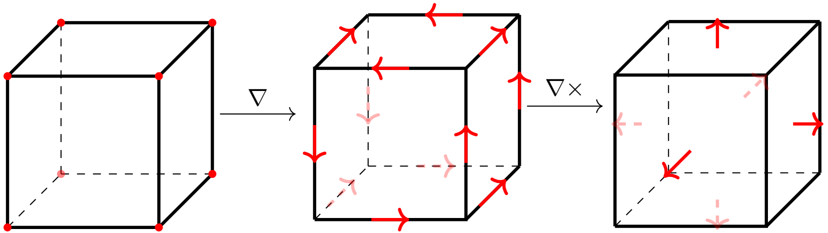


Figure: FEM on blue elements, VEM on red elements

Nodal-edge-face sequence for 3D VEM

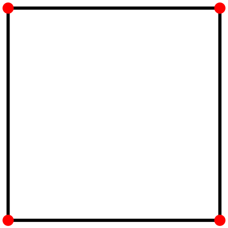
Nodal virtual elements on faces



Nodal virtual elements on faces

Given a face F

$$V_h^{\text{node}}(F) := \left\{ v_h \in \mathcal{C}^0(\overline{F}) \middle| \qquad v_{h|e} \in \mathbb{P}_1(e)\ \forall e \in \mathcal{E}^F, \qquad \right\}$$



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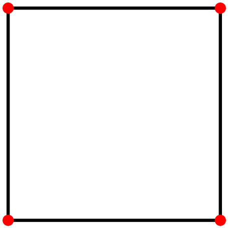

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$$\mathbf{x}_F = \mathbf{x} - \mathbf{b}_F \qquad \forall \mathbf{x} \in F$$

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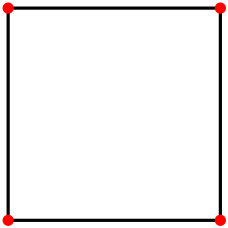

Nodal virtual elements on polyhedra

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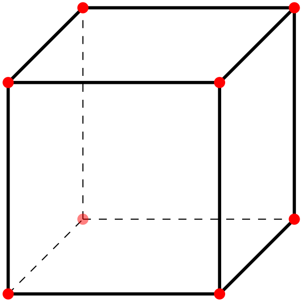

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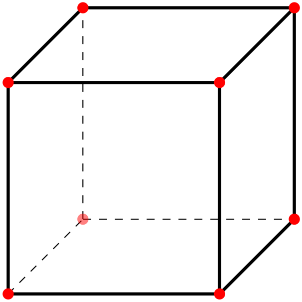
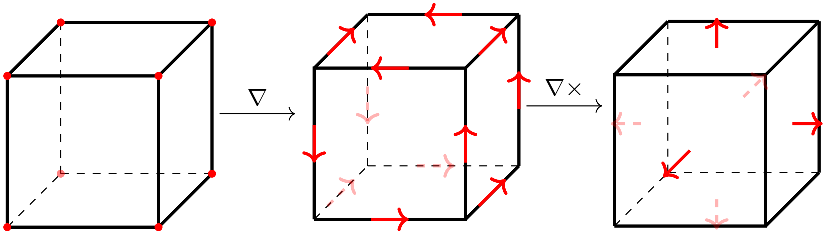


Edge virtual elements on faces



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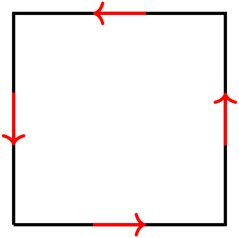
$$\mathbf{V}_h^{\text{edge}}(F) := \left\{ \mathbf{F}_h \in [L^2(F)]^2 \,\middle|\, \mathbf{F}_h \cdot \mathbf{t}_e \in \mathbb{P}_0(e) \ \forall e \in \mathcal{E}^F \right\}$$



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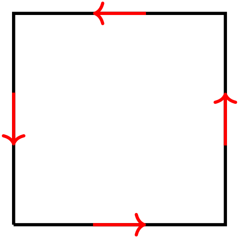

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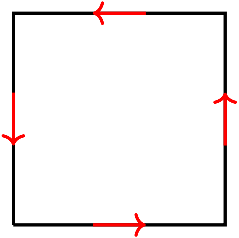

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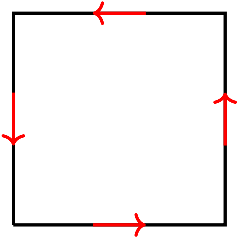

Edge virtual elements on polyhedra

Given $\mathbf{x}_K = \mathbf{x} - \mathbf{b}_K$

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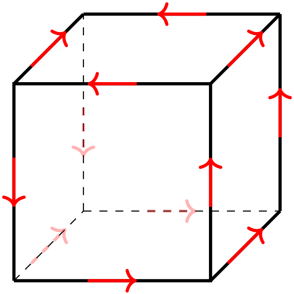

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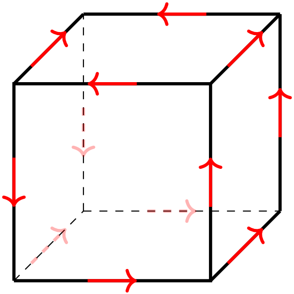
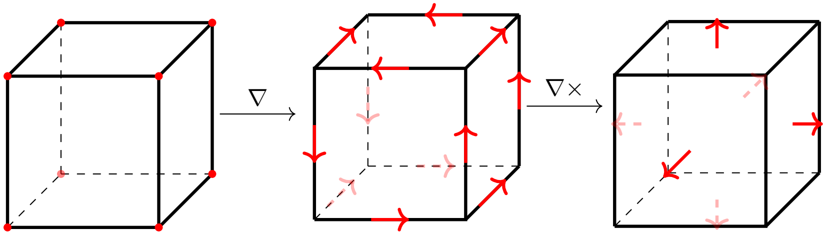


Face virtual elements on polyhedra



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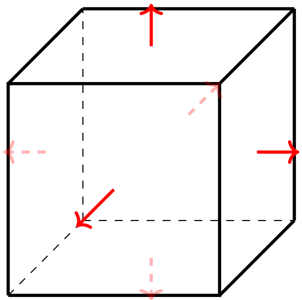

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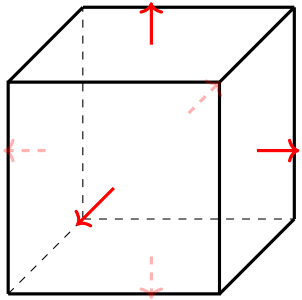

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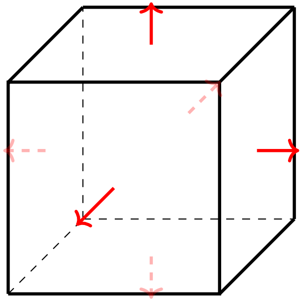

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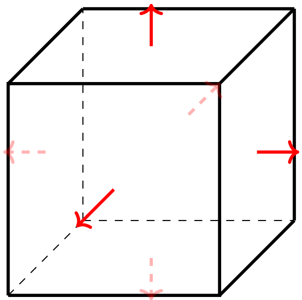

Face virtual elements on polyhedra

$$V_h^{\text{node}}(K) \quad \xrightarrow{\nabla} \quad \mathbf{V}_h^{\text{edge}}(K) \quad \xrightarrow{\nabla\times} \quad \mathbf{V}_h^{\text{face}}(K)$$

Interpolation estimates

Interpolation estimates for finite elements

Finite element interpolation estimates [assuming sufficient regularity]

- map to reference element (standard, Piola, . . .)

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  - polynomial approximation
- map back to physical element
  - milk out scaling

Interpolation estimates for finite elements

For Lagrangian element [up to $K \leftrightarrow \widehat{K}$]

$$\left| v - \mathcal{I}_{FE}^N v \right|_{1,\widehat{K}} \le |v - v_1|_{1,\widehat{K}} + \left| \mathcal{I}_{FE}^N (v - v_1) \right|_{1,\widehat{K}} \lesssim |v - v_1|_{1,\widehat{K}} + \|v - v_1\|_{\frac{3}{2}+\varepsilon,\widehat{K}}$$

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For Raviart-Thomas, similar arguments
[Nédélec, Numer. Math., 1980]

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For Nédélec, similar arguments

[Boffi, Gastaldi, Appl. Numer. Math. 2006], [Amrouche, Bernardi, Dauge, Girault, M2AS, 1998]

Challenges in deriving interpolation estimates in the VEM

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- no reference element is available

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Ways out

- definition of the spaces + integration by parts lead to polynomials
- polynomial inverse estimates on regular subtriangulation of elements are available

Interpolation estimates for face (Raviart-Thomas) elements

FEM – face – L^2 error

$\mathbf{C}$ in $H(div, K) \cap [L^p(K)]^3 \cap [H^s(K)]^3$ with $p > 2$ and $s > 1/2$. Then

$$\left\| \mathbf{C} - \mathcal{I}_{FE}^F \mathbf{C} \right\|_{0,K} \lesssim h_K^s \, |\mathbf{C}|_{s,K}$$

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FEM – face – L^2 div error

C in $H^s(div, K) \cap [L^p(K)]^3$ with $p > 2$ and $s > 0$. Then

$$\left\| \nabla \cdot (\mathbf{C} - \mathcal{I}_{FE}^F \mathbf{C}) \right\|_{0,K} \lesssim h_K^s \, |\nabla \cdot \mathbf{C}|_{s,K}$$

VEM – face – L^2 error

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VEM – face – L^2 div error

C in $H^s(div, K) \cap [L^p(K)]^3$ with $p > 2$ and $s > 0$. Then

$$\left\| \nabla \cdot (\mathbf{C} - \mathcal{I}_{VE}^F \mathbf{C}) \right\|_{0,K} \lesssim h_K^s \, |\nabla \cdot \mathbf{C}|_{s,K}$$

Proof of interpolation estimates for face VE (1)

Let $\mathbf{C}_\pi$ be the vector average of $\mathbf{C}$. Denote $\mathbf{C}_I := \mathcal{I}_{VE}^F \mathbf{C}$ (assign normal components)
Then

$$\|\mathbf{C} - \mathbf{C}_I\|_{0,K} \leq \|\mathbf{C} - \mathbf{C}_\pi\|_{0,K} + \|\mathbf{C}_I - \mathbf{C}_\pi\|_{0,K} \lesssim h_K^s \, |\mathbf{C}|_{s,K} + \|\mathbf{C}_I - \mathbf{C}_\pi\|_{0,K}$$

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As for the second term on the right-hand side, we have

$$\|\mathbf{C}_I-\mathbf{C}_\pi\|_{0,K}$$

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As for the second term on the right-hand side, we have

$$\|\mathbf{C}_I - \mathbf{C}_\pi\|_{0,K} \lesssim h_K^{\frac{1}{2}} \, \|(\mathbf{C}_I - \mathbf{C}_\pi) \cdot \mathbf{n}_K\|_{0,\partial K}$$

HOT POINT

Proof of interpolation estimates for face VE (1)

Let $\mathbf{C}_\pi$ be the vector average of $\mathbf{C}$. Denote $\mathbf{C}_I := \mathcal{I}^F_{VE}\mathbf{C}$ (assign normal components)
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As for the second term on the right-hand side, we have

$\pm\mathbf{C}$

$$\begin{aligned}\|\mathbf{C}_I-\mathbf{C}_\pi\|_{0,K} &\lesssim h_K^{\frac{1}{2}}\,\|(\mathbf{C}_I-\mathbf{C}_\pi)\cdot\mathbf{n}_K\|_{0,\partial K}\\ &\le h_K^{\frac{1}{2}}\,\|(\mathbf{C}-\mathbf{C}_I)\cdot\mathbf{n}_K\|_{0,\partial K} + h_K^{\frac{1}{2}}\,\|(\mathbf{C}-\mathbf{C}_\pi)\cdot\mathbf{n}_K\|_{0,\partial K}\end{aligned}$$

Proof of interpolation estimates for face VE (1)

Let $\mathbf{C}_\pi$ be the vector average of $\mathbf{C}$. Denote $\mathbf{C}_I := \mathcal{I}_{VE}^F \mathbf{C}$ (assign normal components)
Then

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As for the second term on the right-hand side, we have

$$\int_F \mathbf{n}_K \cdot (\mathbf{C} - \mathbf{C}_I) = 0$$

$$\begin{aligned}\|\mathbf{C}_I - \mathbf{C}_\pi\|_{0,K} &\lesssim h_K^{\frac{1}{2}} \, \|(\mathbf{C}_I - \mathbf{C}_\pi) \cdot \mathbf{n}_K\|_{0,\partial K} \\ &\le h_K^{\frac{1}{2}} \, \|(\mathbf{C} - \mathbf{C}_I) \cdot \mathbf{n}_K\|_{0,\partial K} + h_K^{\frac{1}{2}} \, \|(\mathbf{C} - \mathbf{C}_\pi) \cdot \mathbf{n}_K\|_{0,\partial K} \\ &\le 2 h_K^{\frac{1}{2}} \, \|(\mathbf{C} - \mathbf{C}_\pi) \cdot \mathbf{n}_K\|_{0,\partial K}\end{aligned}$$

Proof of interpolation estimates for face VE (1)

Let $\mathbf{C}_\pi$ be the vector average of $\mathbf{C}$. Denote $\mathbf{C}_I := \mathcal{I}_{VE}^F \mathbf{C}$ (assign normal components)
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$$\|\mathbf{C} - \mathbf{C}_I\|_{0,K} \leq \|\mathbf{C} - \mathbf{C}_\pi\|_{0,K} + \|\mathbf{C}_I - \mathbf{C}_\pi\|_{0,K} \lesssim h_K^s \, |\mathbf{C}|_{s,K} + \|\mathbf{C}_I - \mathbf{C}_\pi\|_{0,K}$$

As for the second term on the right-hand side, we have

trace and Poincaré ineq.

$$\begin{aligned} \|\mathbf{C}_I - \mathbf{C}_\pi\|_{0,K} &\lesssim h_K^{\frac{1}{2}} \, \|(\mathbf{C}_I - \mathbf{C}_\pi) \cdot \mathbf{n}_K\|_{0,\partial K} \\ &\leq h_K^{\frac{1}{2}} \, \|(\mathbf{C} - \mathbf{C}_I) \cdot \mathbf{n}_K\|_{0,\partial K} + h_K^{\frac{1}{2}} \, \|(\mathbf{C} - \mathbf{C}_\pi) \cdot \mathbf{n}_K\|_{0,\partial K} \\ &\leq 2 h_K^{\frac{1}{2}} \, \|(\mathbf{C} - \mathbf{C}_\pi) \cdot \mathbf{n}_K\|_{0,\partial K} \lesssim h_K^s \, |\mathbf{C}|_{s,K} \end{aligned}$$

Proof of interpolation estimates for face VE (2)

Hot point: an inverse inequality in the face VE space

$$\|\mathbf{C}_h\|_{0,K} \lesssim h_K^{\frac{1}{2}} \|\mathbf{n}_K \cdot \mathbf{C}_h\|_{0,\partial K}$$

Proof of interpolation estimates for face VE (2)

Hot point: an inverse inequality in the face VE space

$$\|\mathbf{C}_h\|_{0,K} \lesssim h_K^{\frac{1}{2}} \, \|\mathbf{n}_K \cdot \mathbf{C}_h\|_{0,\partial K}$$

We have the Helmholtz decomposition

$$\mathbf{C}_h = \nabla\Psi + \nabla\times\boldsymbol{\rho}$$

where Ψ in $H^1(K) \setminus \mathbb{R}$ and $\boldsymbol{\rho}$ in $H(\nabla\times, K)$ satisfy

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Proof of interpolation estimates for face VE (2)

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Proof of interpolation estimates for face VE (3)

$$(\nabla \times \boldsymbol{\rho}, \nabla \Psi)_{0,K} = 0 \quad \& \quad \text{BCs} \qquad \Longrightarrow \qquad \|\mathbf{C}_h\|_{0,K}^2 = \|\nabla \Psi\|_{0,K}^2 + \|\nabla \times \boldsymbol{\rho}\|_{0,K}^2$$

Proof of interpolation estimates for face VE (3)

$$(\nabla \times \boldsymbol{\rho}, \nabla \Psi)_{0,K} = 0 \quad \& \quad \text{BCs} \qquad \Longrightarrow \qquad \|\mathbf{C}_h\|^2_{0,K} = \|\nabla \Psi\|^2_{0,K} + \|\nabla \times \boldsymbol{\rho}\|^2_{0,K}$$

We estimate the two terms on the right-hand side on the right-hand side

$$\|\nabla \Psi\|^2_{0,K}$$

Proof of interpolation estimates for face VE (3)

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We estimate the two terms on the right-hand side on the rig

IBP + definition of Ψ

$$\|\nabla\Psi\|^2_{0,K} = -\int_K \nabla\cdot \mathbf{C}_h \, \Psi + \int_{\partial K} \mathbf{n}_K \cdot \mathbf{C}_h \, \Psi$$

Proof of interpolation estimates for face VE (3)

$$(\nabla\times \boldsymbol{\rho}, \nabla\Psi)_{0,K} = 0 \quad \& \quad \text{BCs} \qquad \Longrightarrow \qquad \|\mathbf{C}_h\|^2_{0,K} = \|\nabla\Psi\|^2_{0,K} + \|\nabla\times \boldsymbol{\rho}\|^2_{0,K}$$

We estimate the two terms on the right-hand side on the

$\nabla\cdot \mathbf{C}_h \in \mathbb{R}$, Ψ zero average

$$\|\nabla\Psi\|^2_{0,K} = -\int_K \nabla\cdot \mathbf{C}_h\, \Psi + \int_{\partial K} \mathbf{n}_K \cdot \mathbf{C}_h\, \Psi = \int_{\partial K} \mathbf{n}_K \cdot \mathbf{C}_h\, \Psi$$

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$$\leq \|\mathbf{n}_K \cdot \mathbf{C}_h\|_{0,\partial K} \, \|\Psi\|_{0,\partial K}$$

Proof of interpolation estimates for face VE (3)

$$(\nabla\times \boldsymbol{\rho}, \nabla\Psi)_{0,K} = 0 \quad \& \quad \text{BCs} \qquad \Longrightarrow \qquad \|\mathbf{C}_h\|_{0,K}^2 = \|\nabla\Psi\|_{0,K}^2 + \|\nabla\times \boldsymbol{\rho}\|_{0,K}^2$$

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$$\|\nabla\Psi\|_{0,K}^2 = -\int_K \nabla\cdot \mathbf{C}_h \, \Psi + \int_{\partial K} \mathbf{n}_K \cdot \mathbf{C}_h \, \Psi = \int_{\partial K} \mathbf{n}_K \cdot \mathbf{C}_h \, \Psi$$
$$\leq \|\mathbf{n}_K \cdot \mathbf{C}_h\|_{0,\partial K} \, \|\Psi\|_{0,\partial K} \lesssim \|\mathbf{n}_K \cdot \mathbf{C}_h\|_{0,\partial K} \, h_K^{\frac{1}{2}} \, \|\nabla\Psi\|_{0,K}$$

Proof of interpolation estimates for face VE (3)

$$(\nabla\times \boldsymbol{\rho}, \nabla\Psi)_{0,K} = 0 \quad \& \quad \text{BCs} \qquad \Longrightarrow \qquad \|\mathbf{C}_h\|^2_{0,K} = \|\nabla\Psi\|^2_{0,K} + \|\nabla\times \boldsymbol{\rho}\|^2_{0,K}$$

We estimate the two terms on the right-hand side on the right-hand side

$$\|\nabla\Psi\|^2_{0,K} = -\int_K \nabla\cdot \mathbf{C}_h \, \Psi + \int_{\partial K} \mathbf{n}_K \cdot \mathbf{C}_h \, \Psi = \int_{\partial K} \mathbf{n}_K \cdot \mathbf{C}_h \, \Psi$$
$$\leq \|\mathbf{n}_K \cdot \mathbf{C}_h\|_{0,\partial K} \, \|\Psi\|_{0,\partial K} \lesssim \|\mathbf{n}_K \cdot \mathbf{C}_h\|_{0,\partial K} \, h_K^{\frac{1}{2}} \, \|\nabla\Psi\|_{0,K}$$

"only" direct estimates are used

We end up with

$$\|\nabla\Psi\|^2_{0,K} \lesssim h_K \, \|\mathbf{n}_K \cdot \mathbf{C}_h\|^2_{0,\partial K}$$

Proof of interpolation estimates for face VE (4)

$$\|\nabla\times \boldsymbol{\rho}\|_{0,K}^2$$

Proof of interpolation estimates for face VE (4)

IBP and $\mathbf{n}_K \times \boldsymbol{\rho} = \mathbf{0}$ on ∂K

$$\|\nabla\times\boldsymbol{\rho}\|_{0,K}^2 = \int_K \boldsymbol{\rho}\cdot\nabla\times\nabla\times\boldsymbol{\rho}$$

Proof of interpolation estimates for face VE (4)

$\nabla \times \nabla \times \boldsymbol{\rho} = \nabla \times \mathbf{C}_h$

$$\|\nabla \times \boldsymbol{\rho}\|_{0,K}^2 = \int_K \boldsymbol{\rho} \cdot \nabla \times \nabla \times \boldsymbol{\rho} = \int_K \boldsymbol{\rho} \cdot \nabla \times \mathbf{C}_h$$

Proof of interpolation estimates for face VE (4)

$$\|\nabla\times \boldsymbol{\rho}\|_{0,K}^2 = \int_K \boldsymbol{\rho}\cdot\nabla\times\nabla\times\boldsymbol{\rho} = \int_K \boldsymbol{\rho}\cdot\nabla\times\mathbf{C}_h$$

Set $\mathbf{q}_0 := \nabla\times\mathbf{C}_h \in [\mathbb{P}_0(K)]^3$. We have

direct computation

$$\mathbf{q}_0 = \frac{1}{2}\nabla\times(\mathbf{q}_0\times\mathbf{x}_K)$$

Then

Proof of interpolation estimates for face VE (4)

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$$\|\nabla\times \boldsymbol{\rho}\|_{0,K}^2$$

Proof of interpolation estimates for face VE (4)

$$\|\nabla\times \boldsymbol{\rho}\|^2_{0,K} = \int_K \boldsymbol{\rho}\cdot\nabla\times\nabla\times\boldsymbol{\rho} = \int_K \boldsymbol{\rho}\cdot\nabla\times\mathbf{C}_h$$

Set $\mathbf{q}_0 := \nabla\times\mathbf{C}_h \in [\mathbb{P}_0(K)]^3$. We have

$\nabla\times\mathbf{C}_h = \frac{1}{2}\nabla\times(\mathbf{q}_0\times\mathbf{x}_K)$

$$\mathbf{q}_0 = \frac{1}{2}\nabla\times(\mathbf{q}_0\times\mathbf{x}_K)$$

Then

$$\|\nabla\times \boldsymbol{\rho}\|^2_{0,K} = \frac{1}{2}\int_K \boldsymbol{\rho}\cdot\nabla\times(\mathbf{q}_0\times\mathbf{x}_K)$$

Proof of interpolation estimates for face VE (4)

$$\|\nabla\times \boldsymbol{\rho}\|^2_{0,K} = \int_K \boldsymbol{\rho}\cdot\nabla\times\nabla\times\boldsymbol{\rho} = \int_K \boldsymbol{\rho}\cdot\nabla\times\mathbf{C}_h$$

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$$\mathbf{q}_0 = \frac{1}{2}\nabla\times(\mathbf{q}_0\times\mathbf{x}_K)$$

Then

$$\begin{aligned}\|\nabla\times \boldsymbol{\rho}\|^2_{0,K} &= \frac{1}{2}\int_K \boldsymbol{\rho}\cdot\nabla\times(\mathbf{q}_0\times\mathbf{x}_K)\\ &= \frac{1}{2}\int_K \nabla\times\boldsymbol{\rho}\cdot(\mathbf{q}_0\times\mathbf{x}_K) + \frac{1}{2}\int_{\partial K}(\mathbf{n}_K\times\boldsymbol{\rho})\cdot(\mathbf{q}_0\times\mathbf{x}_K)\end{aligned}$$

IBP

Proof of interpolation estimates for face VE (4)

$$\|\nabla\times\boldsymbol{\rho}\|^2_{0,K} = \int_K \boldsymbol{\rho}\cdot\nabla\times\nabla\times\boldsymbol{\rho} = \int_K \boldsymbol{\rho}\cdot\nabla\times\mathbf{C}_h$$

Set $\mathbf{q}_0 := \nabla\times\mathbf{C}_h \in [\mathbb{P}_0(K)]^3$. We have

$$\mathbf{q}_0 = \frac{1}{2}\nabla\times(\mathbf{q}_0\times\mathbf{x}_K)$$

Then

$$\begin{aligned}\|\nabla\times\boldsymbol{\rho}\|^2_{0,K} &= \frac{1}{2}\int_K \boldsymbol{\rho}\cdot\nabla\times(\mathbf{q}_0\times\mathbf{x}_K)\\ &= \frac{1}{2}\int_K \nabla\times\boldsymbol{\rho}\cdot(\mathbf{q}_0\times\mathbf{x}_K) + \frac{1}{2}\int_{\partial K}(\mathbf{n}_K\times\boldsymbol{\rho})\cdot(\mathbf{q}_0\times\mathbf{x}_K)\\ &= \frac{1}{2}\int_K \nabla\times\boldsymbol{\rho}\cdot(\mathbf{q}_0\times\mathbf{x}_K)\end{aligned}$$

$\mathbf{n}_K\times\boldsymbol{\rho} = \mathbf{0}$

Proof of interpolation estimates for face VE (4)

$$\|\nabla\times \boldsymbol{\rho}\|^2_{0,K} = \int_K \boldsymbol{\rho}\cdot\nabla\times\nabla\times\boldsymbol{\rho} = \int_K \boldsymbol{\rho}\cdot\nabla\times\mathbf{C}_h$$

Set $\mathbf{q}_0 := \nabla\times\mathbf{C}_h \in [\mathbb{P}_0(K)]^3$. We have

$$\mathbf{q}_0 = \frac{1}{2}\nabla\times(\mathbf{q}_0\times\mathbf{x}_K)$$

Then

$$\begin{aligned}\|\nabla\times \boldsymbol{\rho}\|^2_{0,K} &= \frac{1}{2}\int_K \boldsymbol{\rho}\cdot\nabla\times(\mathbf{q}_0\times\mathbf{x}_K)\\ &= \frac{1}{2}\int_K \nabla\times\boldsymbol{\rho}\cdot(\mathbf{q}_0\times\mathbf{x}_K) + \frac{1}{2}\int_{\partial K}(\mathbf{n}_K\times\boldsymbol{\rho})\cdot(\mathbf{q}_0\times\mathbf{x}_K)\\ &= \frac{1}{2}\int_K \nabla\times\boldsymbol{\rho}\cdot(\mathbf{q}_0\times\mathbf{x}_K) = \frac{1}{2}\int_K(\mathbf{C}_h-\nabla\Psi)\cdot(\mathbf{q}_0\times\mathbf{x}_K)\end{aligned}$$

$\mathbf{C}_h = \nabla\Psi + \nabla\times\boldsymbol{\rho}$

Proof of interpolation estimates for face VE (4)

$$\|\nabla\times \boldsymbol{\rho}\|^2_{0,K} = \int_K \boldsymbol{\rho}\cdot\nabla\times\nabla\times\boldsymbol{\rho} = \int_K \boldsymbol{\rho}\cdot\nabla\times\mathbf{C}_h$$

Set $\mathbf{q}_0 := \nabla\times\mathbf{C}_h \in [\mathbb{P}_0(K)]^3$. We have

$$\mathbf{q}_0 = \frac{1}{2}\nabla\times(\mathbf{q}_0\times\mathbf{x}_K)$$

Then

$$\begin{aligned}\|\nabla\times\boldsymbol{\rho}\|^2_{0,K} &= \frac{1}{2}\int_K \boldsymbol{\rho}\cdot\nabla\times(\mathbf{q}_0\times\mathbf{x}_K)\\ &= \frac{1}{2}\int_K \nabla\times\boldsymbol{\rho}\cdot(\mathbf{q}_0\times\mathbf{x}_K) + \frac{1}{2}\int_{\partial K}(\mathbf{n}_K\times\boldsymbol{\rho})\cdot(\mathbf{q}_0\times\mathbf{x}_K)\\ &= \frac{1}{2}\int_K \nabla\times\boldsymbol{\rho}\cdot(\mathbf{q}_0\times\mathbf{x}_K) = \frac{1}{2}\int_K(\mathbf{C}_h-\nabla\Psi)\cdot(\mathbf{q}_0\times\mathbf{x}_K)\\ &= -\frac{1}{2}\int_K \nabla\Psi\cdot(\mathbf{q}_0\times\mathbf{x}_K)\end{aligned}$$

$\int_K \mathbf{C}_h\cdot(\mathbf{q}_0\times\mathbf{x}_K) = 0$

Proof of interpolation estimates for face VE (4)

$$\|\nabla\times\boldsymbol{\rho}\|_{0,K}^2 = \int_K \boldsymbol{\rho}\cdot\nabla\times\nabla\times\boldsymbol{\rho} = \int_K \boldsymbol{\rho}\cdot\nabla\times\mathbf{C}_h$$

Set $\mathbf{q}_0 := \nabla\times\mathbf{C}_h \in [\mathbb{P}_0(K)]^3$. We have

$$\mathbf{q}_0 = \frac{1}{2}\nabla\times(\mathbf{q}_0\times\mathbf{x}_K)$$

Then

$$\begin{aligned}\|\nabla\times\boldsymbol{\rho}\|_{0,K}^2 &= \frac{1}{2}\int_K \boldsymbol{\rho}\cdot\nabla\times(\mathbf{q}_0\times\mathbf{x}_K)\\ &= \frac{1}{2}\int_K \nabla\times\boldsymbol{\rho}\cdot(\mathbf{q}_0\times\mathbf{x}_K) + \frac{1}{2}\int_{\partial K}(\mathbf{n}_K\times\boldsymbol{\rho})\cdot(\mathbf{q}_0\times\mathbf{x}_K)\\ &= \frac{1}{2}\int_K \nabla\times\boldsymbol{\rho}\cdot(\mathbf{q}_0\times\mathbf{x}_K) = \frac{1}{2}\int_K(\mathbf{C}_h-\nabla\Psi)\cdot(\mathbf{q}_0\times\mathbf{x}_K)\\ &= -\frac{1}{2}\int_K \nabla\Psi\cdot(\mathbf{q}_0\times\mathbf{x}_K) \le \frac{1}{2}h_K\,|\Psi|_{1,K}\,\|\mathbf{q}_0\|_{0,K}\end{aligned}$$

$\|\mathbf{x}_K\|_{L^\infty} \le h_K$

Proof of interpolation estimates for face VE (5)

Since

$$\mathbf{q}_0 = \nabla\times \mathbf{C}_h = \nabla\times \nabla\times \boldsymbol{\rho}$$

Proof of interpolation estimates for face VE (5)

Since

$$\mathbf{q}_0 = \nabla\times \mathbf{C}_h = \nabla\times \nabla\times \boldsymbol{\rho}$$

we end up with

$$\|\nabla\times \boldsymbol{\rho}\|_{0,K}^2 \le \frac{1}{2} h_K \, |\Psi|_{1,K} \, \|\nabla\times \nabla\times \boldsymbol{\rho}\|_{0,K}$$

Proof of interpolation estimates for face VE (5)

Since

$$\mathbf{q}_0 = \nabla \times \mathbf{C}_h = \nabla \times \nabla \times \boldsymbol{\rho}$$

we end up with

$$\|\nabla \times \boldsymbol{\rho}\|^2_{0,K} \leq \frac{1}{2} h_K \, |\Psi|_{1,K} \, \|\nabla \times \nabla \times \boldsymbol{\rho}\|_{0,K}$$

If we were able to show

$$h_K \, \|\nabla \times \nabla \times \boldsymbol{\rho}\|_{0,K} \lesssim \|\nabla \times \boldsymbol{\rho}\|_{0,K}$$

Proof of interpolation estimates for face VE (5)

Since

$$\mathbf{q}_0 = \nabla\times \mathbf{C}_h = \nabla\times \nabla\times \boldsymbol{\rho}$$

we end up with

$$\|\nabla\times \boldsymbol{\rho}\|^2_{0,K} \le \frac{1}{2} h_K \, |\Psi|_{1,K} \, \|\nabla\times \nabla\times \boldsymbol{\rho}\|_{0,K}$$

If we were able to show

$$h_K \, \|\nabla\times \nabla\times \boldsymbol{\rho}\|_{0,K} \lesssim \|\nabla\times \boldsymbol{\rho}\|_{0,K}$$

then we would conclude with

$$\|\nabla\times \boldsymbol{\rho}\|_{0,K} \lesssim |\Psi|_{1,K}$$

Proof of interpolation estimates for face VE (5)

Since

$$\mathbf{q}_0 = \nabla \times \mathbf{C}_h = \nabla \times \nabla \times \boldsymbol{\rho}$$

we end up with

$$\|\nabla \times \boldsymbol{\rho}\|_{0,K}^2 \leq \frac{1}{2} h_K \, |\Psi|_{1,K} \, \|\nabla \times \nabla \times \boldsymbol{\rho}\|_{0,K}$$

If we were able to show

$$h_K \, \|\nabla \times \nabla \times \boldsymbol{\rho}\|_{0,K} \lesssim \|\nabla \times \boldsymbol{\rho}\|_{0,K}$$

then we would conclude with

estimates on $\nabla\Psi$

$$\|\nabla \times \boldsymbol{\rho}\|_{0,K} \lesssim |\Psi|_{1,K} \lesssim h_K^{\frac{1}{2}} \, \|\mathbf{n}_K \cdot \mathbf{C}_h\|_{0,\partial K}$$

Proof of interpolation estimates for face VE (6)

Thus, we prove

$$h_K \, \|\nabla\times \nabla\times \boldsymbol{\rho}\|_{0,K} \lesssim \|\nabla\times \boldsymbol{\rho}\|_{0,K}$$

Proof of interpolation estimates for face VE (6)

Thus, we prove

$$h_K \|\nabla \times \nabla \times \boldsymbol{\rho}\|_{0,K} \lesssim \|\nabla \times \boldsymbol{\rho}\|_{0,K}$$

Let b_K denote the piecewise quadratic bubble over a subtriangulation of K

Proof of interpolation estimates for face VE (6)

Thus, we prove

$$h_K \, \|\nabla\times \nabla\times \boldsymbol{\rho}\|_{0,K} \lesssim \|\nabla\times \boldsymbol{\rho}\|_{0,K}$$

Let b_K denote the piecewise quadratic bubble over a subtriangulation of K

$$\|\nabla\times \nabla\times \boldsymbol{\rho}\|_{0,K}^2$$

Proof of interpolation estimates for face VE (6)

Thus, we prove

$$h_K \, \|\nabla\times\nabla\times\boldsymbol{\rho}\|_{0,K} \lesssim \|\nabla\times\boldsymbol{\rho}\|_{0,K}$$

Let b_K denote the piecewise quadratic bubble over a subtriangulation of K

$$\|\nabla\times\nabla\times\boldsymbol{\rho}\|_{0,K}^2 \approx \int_K b_K(\nabla\times\nabla\times\boldsymbol{\rho})^2$$

polyn. inverse est.

Proof of interpolation estimates for face VE (6)

Thus, we prove

$$h_K \, \|\nabla\times \nabla\times \boldsymbol{\rho}\|_{0,K} \lesssim \|\nabla\times \boldsymbol{\rho}\|_{0,K}$$

Let b_K denote the piecewise quadratic bubble over a subtriangulation of K

$$\|\nabla\times \nabla\times \boldsymbol{\rho}\|^2_{0,K} \approx \int_K b_K (\nabla\times \nabla\times \boldsymbol{\rho})^2 = \int_K \nabla\times(b_K \nabla\times \nabla\times \boldsymbol{\rho}) \cdot \nabla\times \boldsymbol{\rho}$$

IBP + $b_{K|\partial K} = 0$

Proof of interpolation estimates for face VE (6)

Thus, we prove

$$h_K \, \|\nabla\times\nabla\times\boldsymbol{\rho}\|_{0,K} \lesssim \|\nabla\times\boldsymbol{\rho}\|_{0,K}$$

Let b_K denote the piecewise quadratic bubble over a subtriangulation of K

$$\begin{aligned}\|\nabla\times\nabla\times\boldsymbol{\rho}\|_{0,K}^2 &\approx \int_K b_K(\nabla\times\nabla\times\boldsymbol{\rho})^2 = \int_K \nabla\times(b_K\nabla\times\nabla\times\boldsymbol{\rho})\cdot\nabla\times\boldsymbol{\rho} \\ &\leq \|\nabla\times(b_K\nabla\times\nabla\times\boldsymbol{\rho})\|_{0,K}\,\|\nabla\times\boldsymbol{\rho}\|_{0,K}\end{aligned}$$

Proof of interpolation estimates for face VE (6)

Thus, we prove

$$h_K \, \|\nabla\times\nabla\times \boldsymbol{\rho}\|_{0,K} \lesssim \|\nabla\times \boldsymbol{\rho}\|_{0,K}$$

Let b_K denote the piecewise quadratic bubble over a subtriangulation of K

$$\begin{aligned}\|\nabla\times\nabla\times \boldsymbol{\rho}\|_{0,K}^2 &\approx \int_K b_K (\nabla\times\nabla\times \boldsymbol{\rho})^2 = \int_K \nabla\times(b_K \nabla\times\nabla\times \boldsymbol{\rho}) \cdot \nabla\times \boldsymbol{\rho} \\ &\leq \|\nabla\times(b_K \nabla\times\nabla\times \boldsymbol{\rho})\|_{0,K} \, \|\nabla\times \boldsymbol{\rho}\|_{0,K} \\ &\lesssim h_K^{-1} \, \|\nabla\times\nabla\times \boldsymbol{\rho}\|_{0,K} \, \|\nabla\times \boldsymbol{\rho}\|_{0,K}\end{aligned}$$

polyn. inverse est., $\|b_K\|_{L^\infty(K)} \approx 1$

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$$\begin{aligned}
\|\nabla\times\nabla\times \boldsymbol{\rho}\|_{0,K}^2 \approx \int_K b_K (\nabla\times\nabla\times\boldsymbol{\rho})^2 &= \int_K \nabla\times(b_K\,\nabla\times\nabla\times\boldsymbol{\rho})\cdot\nabla\times\boldsymbol{\rho} \\
&\leq \|\nabla\times(b_K\,\nabla\times\nabla\times\boldsymbol{\rho})\|_{0,K}\,\|\nabla\times\boldsymbol{\rho}\|_{0,K} \\
&\lesssim h_K^{-1}\,\|\nabla\times\nabla\times\boldsymbol{\rho}\|_{0,K}\,\|\nabla\times\boldsymbol{\rho}\|_{0,K}
\end{aligned}$$

Estimates in the divergence norm are trivial ($\nabla\cdot\mathbf{C}_I$ is the average of $\nabla\cdot\mathbf{C}$)

Related results

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- review and general techniques
  [Mascotto, CAMWA, 2023]

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- VEM exact sequence [$\leftrightarrow$ FEM exact sequence]
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- interpolation estimates are explicit in the geometry of the elements
- stability estimates are proven using similar tools

Thank you!

Related aspects: stability estimates

Given a virtual element space $V_h(K)$, one needs stability estimates of the form

$$\alpha_* \, |v_h|^2_{?,K} \leq S^K(v_h, v_h) \leq \alpha^* \, |v_h|^2_{?,K} \qquad \forall v_h \in V_h(K) \cap \ker(\Pi^?)$$

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For instance, we want to prove for face VE spaces

$$\alpha_* \, \|\mathbf{C}_h\|^2_{0,K} \leq S^K(\mathbf{C}_h, \mathbf{C}_h) \leq \alpha^* \, \|\mathbf{C}_h\|^2_{0,K} \qquad \forall \mathbf{C}_h \in \mathbf{V}_h^{\text{face}}(K)) \cap [\ker(\Pi^0)]^3$$

where

$$S^K(\mathbf{E}_h, \mathbf{C}_h) := h_K \sum_{F \in \mathcal{F}_h} (\mathbf{n}_F \cdot \mathbf{E}_h, \mathbf{n}_F \cdot \mathbf{C}_h)_{0,\partial K}$$

Stability estimates for face VE elements

The lower bound is an immediate consequence of the already proven inequality

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polyn. inverse est.

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$H(\nabla\cdot)$ trace ineq.

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In fact, we could also prove

$$\|\nabla\cdot \mathbf{C}_h\|_{0,K} \lesssim h_K^{-1} \|\mathbf{C}_h\|_{0,K}$$

which gives the upper bound

Interpolation estimates for nodal (Lagrangian) elements

FEM – nodal

v in $H^s(K)$, $3/2 < s \leq 2$. Then

$$\left| v - \mathcal{I}_{FE}^N v \right|_{1,K} \lesssim h_K^s \, |v|_{s,K}$$

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Interpolation estimates for edge (Nédélec) elements

FEM – edge (high regularity)

F in $[H^s(K)]^3$ with $s > 1$, then

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FEM – edge [Boffi, Gastaldi, Appl. Numer. Math. 2006]

F in $[H^s(K)]^3$, $s \in (1/2, 1]$, with $\nabla\times$ **F** in $L^p(K)$, $p > 2$, then

$$\left\|\mathbf{F} - \mathcal{I}_{FE}^{E}\mathbf{F}\right\|_{0,K} \lesssim h_K^s (|\mathbf{F}|_{s,K} + \|\nabla\times\mathbf{F}\|_{L^p(K)})$$

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FEM – edge [Boffi, Gastaldi, Appl. Numer. Math. 2006]

If we further have $\nabla\times \mathbf{F}$ in $H^s(K)$, $0 < s < 1$, then

$$\left\| \nabla\times (\mathbf{F} - \mathcal{I}_{FE}^{E} \mathbf{F}) \right\|_{0,K} \lesssim h_K^s \, |\nabla\times \mathbf{F}|_{s,K}$$

Interpolation estimates for edge (Nédélec) elements

VEM – edge

F in $H^s(\nabla\times, K)$, $1/2 < s \leq 1$, such that $\mathbf{F}_{|e} \cdot \mathbf{t}_e$ in $L^1(e)$. Then

$$\left\|\mathbf{F} - \mathcal{I}_{VE}^{E}\mathbf{F}\right\|_{0,K} + \left\|\nabla\times(\mathbf{F} - \mathcal{I}_{VE}^{E}\mathbf{F})\right\|_{0,K} \lesssim h_K^s \left|\mathbf{F}\right|_{s,K} + h_K \left\|\nabla\times \mathbf{F}\right\|_{0,K} + h_K^s \left|\nabla\times \mathbf{F}\right|_{s,K}$$